54
+50
104

Test 2 In-class Spring 2017 Name _____

Show all necessary work for full credit. You may use your calculator.

1. A) Solve y'' – 12y' + 61y = 0.

3

$\lambda^2 - 12\lambda + 61 = 0$

$\lambda = 6 \pm 5i$

$y(x) = e^{6x}(c_1 \cos(5x) + c_2 \sin(5x))$

B) Solve y'' + 8y' – 9y = 0.

3

$\lambda^2 + 8\lambda - 9 = 0$

$(\lambda + 9)(\lambda - 1) = 0$

$\lambda = -9 \quad \lambda = 1$

$y(x) = c_1 e^{-9x} + c_2 e^{x}$

C) Solve y'' + 12y' + 36y = 0.

3

$\lambda^2 + 12\lambda + 36 = 0$

$(\lambda + 6)^2 = 0$

$\lambda = -6$

$y(x) = c_1 e^{-6x} + c_2 x e^{-6x}$

3 D) Solve y''' – 5y'' = 0.

$\lambda^3 - 5\lambda^2 = 0$

$\lambda^2(\lambda - 5) = 0$

$\lambda = 0 \quad \lambda = 5$

$y = c_1 + c_2 x + c_3 e^{5x}$

2. Determine the form of the particular solution, $y_p$, to $L(y) = \phi(x)$ for the given $\phi(x)$ if the solution to the homogeneous equation $L(y) = 0$ is $y_h = c_1 + c_2 e^{4x} + c_3 e^{-x}$.

3 A) $\phi(x) = 3e^{2x} + 7e^{4x}$

$y_p = Ae^{2x} + Bxe^{4x}$

3 B) $\phi(x) = 2\cos(8x)$

$y_p = A\cos(8x) + B\sin(8x)$

4 C) $\phi(x) = xe^{5x} + 4x - 9$

$y_p = (A_1 x + A_0)e^{5x} + A_2 x^2 + A_3 x$

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3. A complete set of roots for the characteristic equation of an 5th-order homogeneous differential equation in y(x) with real numbers as coefficients is 6, $5\pm i$, $1\pm\sqrt{2}$. Determine the general solution of the differential equation.

5

$$y(x) = c_1 e^{6x} + e^{5x}\left(c_2 \cos(x) + c_3 \sin(x)\right) + c_4 e^{(1+\sqrt{2})x} + c_5 e^{(1-\sqrt{2})x}$$

4. **Set up** the intial-value problems for the following scenarios (give the differential equation and initial conditions). Do not solve.

A) An RC circuit has an emf (in volts) given by 3 sin(2t), a resistance of 10 ohms, and capacitance of 1/20 farad, and no initial charge on the capacitor. Write the differential equation in terms of the charge.

4

$$10q' + 20q = 3\sin(2t) \qquad q(0) = 0$$

B) An RCL circuit connected in series with a resistance of 180 ohms, a capacitor of 1/280 farad, and an inductance of 10 henries has an applied voltage of 12 volts. The initial charge is 1/10 coulomb and the initial current is 1 ampere.

4

$$10q'' + 180q' + 280q = 12 \qquad q(0) = \tfrac{1}{10}$$

$$q'(0) = i(0) = 1$$

C) A mass of ½ slug is attached to a spring which stretches the spring 1.28 ft from it natural length. The mass is started in motion from the equilibrium position with an initial velocity of 4 ft/sec in the upward direction. Assume that the damping force due to air resistance is 3 times the instantaneous velocity.

4

$$\tfrac{1}{2}(32) = k(1.28)$$

$$k = 12.5$$

$$\tfrac{1}{2}x'' + 3x' + 12.5x = 0$$

$$x(0) = 0$$

$$x'(0) = -4$$

D) An 128 lb weight is attached to a spring having a spring constant of 64 lb/ft. The weight is started into motion with no initial velocity by displacing it 9 inches below the equilibrium position and by applying an external force of F(t) = 8 sin(4t). Assume the surrounding medium offers a negligible resistance.

4

$$m = \frac{128}{32} = 4 \qquad 4x'' + 64x = 8\sin(4t)$$

$$k = 64$$

$$x(0) = \frac{3}{4}$$

$$x'(0) = 0$$

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6 5. Solve $y'' - y' - 2y = 10\sin(2x)$ ; $y(0) = 1$ and $y'(0) = 0$ using the method of undetermined coefficients.

$\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0$

$\lambda = 2 \quad \lambda = -1$

$y_H = c_1 e^{2x} + c_2 e^{-x}$

$y_p = A\sin(2x) + B\cos(2x)$

$y_p' = 2A\cos(2x) - 2B\sin(2x)$

$y_p'' = -4A\sin(2x) - 4B\cos(2x)$

$-4A\sin(2x) - 4B\cos(2x) - 2A\cos(2x) + 2B\sin(2x) - 2A\sin(2x) - 2B\cos(2x) = 10\sin(2x)$

$-6A + 2B = 10$

$-2A - 6B = 0$

$-18A + 6B = 30$

$-2A - 6B = 0$

$-20A = 30$

$A = -\frac{3}{2}$

$B = \frac{1}{2}$

$y = c_1 e^{2x} + c_2 e^{-x} - \frac{3}{2}\sin(2x) + \frac{1}{2}\cos(2x)$

$y(0) = 1 : \quad 1 = c_1 + c_2 + \frac{1}{2} \qquad c_1 + c_2 = \frac{1}{2}$

$y'(0) = 0 = \quad 0 = 2c_1 - c_2 - 3 \qquad 2c_1 - c_2 = 3$

$3c_1 = \frac{7}{2} \quad c_1 = \frac{7}{6}$

$c_2 = -2/3$

$$\boxed{y = \frac{7}{6}e^{2x} - \frac{2}{3}e^{-x} - \frac{3}{2}\sin(2x) + \frac{1}{2}\cos(2x)}$$

6 6. Solve $2y'' - 4y' + 2y = \frac{6e^x}{x}$ using variation of parameters.

$y'' - 2y' + y = \frac{3e^x}{x}$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1$

$y_H = c_1 e^x + c_2 x e^x$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = xe^{2x} + e^{2x} - xe^{2x} = e^{2x}$$

$$W_1 = \begin{vmatrix} 0 & xe^x \\ \frac{3e^x}{x} & xe^x + e^x \end{vmatrix} = -3e^{2x}$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{3e^x}{x} \end{vmatrix} = \frac{3e^{2x}}{x}$$

$$u_1' = \frac{-3e^{2x}}{e^{2x}} = -3$$

$u_1 = -3x$

$$u_2' = \frac{\frac{3e^{2x}}{x}}{e^{2x}} = \frac{3}{x}$$

$u_2 = 3\ln|x|$

$y = c_1 e^x + c_2 x e^x - 3xe^x + 3\ln x \, xe^x$

Test 2 Take-home part Name _____

This should be an individual effort. You may use your book, calculator and notes to complete this. Using the internet, tutors, friends, etc is considered cheating. Please do all problems neatly and in order on your own paper. Show all necessary work for full credit. This is due at the beginning of class on March 8. No late papers will be accepted.

1. Solve the IVP using the method of undetermined coefficients: $y''' - 2y'' + 5y' = e^{4x} + 2x - 7$ ; $y(0) = 0$;

   $y'(0) = 0$; $y''(0) = 0$.

2. A spring mass system has a spring constant of 3 lb/ft. A 64-lb weight is attached to the spring. Assume that a damping force numerically equal to 4 times the instantaneous velocity acts on the system. The mass is released from an initial position of 4 inches below the equilibrium position and given an upward velocity of 2 ft/sec.

   A) Find the equation of motion of the spring.

   B) How many times does the mass pass through equilibrium?

3. If the differential equation for a spring-mass system is $Ax'' + 2x' + 9x = 0$,

   A) For what values of A is the system overdamped?

   B) For what values of A is the system critically damped?

4. Given the equation of motion $x(t) = \sin(4t) - \sqrt{3}\cos(4t)$:

   A) Write the equation of motion in the form $x(t) = A\sin(\omega t + \phi)$.

   B) What is the period of motion?

   C) What is the amplitude?

   D) At what time does the mass first pass through the equilibrium position?

5. A 15-volt electromotive force is applied to an LR-series circuit in which the inductance is 0.3 henries and the resistance is 30 ohms.

   A) Find the current $i(t)$ if $i(0) = 0$.

   B) Determine the current as $t \to \infty$.

6. Solve the DE $2y'' + 8y' + 8y = e^{-2t}\ln(t)$ using variation of parameters subject to the initial conditions $y(1) = 2$ and

   $y'(1) = -4$.

7. An RLC circuit connected in series with a resistance of 2 ohms, a condenser of capacitance of 1/13 farad, and an inductance of 1/4 henry has an applied emf $E(t) = 40\cos(2t)$ volts. Assuming no initial current and an initial charge on the capacitor of 3 coulombs, find the charge on the capacitor.

1. $y''' - 2y'' + 5y' = e^{4x} + 2x - 7 \quad y(0)=0 \quad y'(0)=0 \quad y''(0)=0$

$\lambda^3 - 2\lambda^2 + 5\lambda = 0$

$\lambda(\lambda^2 - 2\lambda + 5) = 0$

$\lambda = 0 \quad \lambda = 1 \pm 2i$

$y_H = c_1 + c_2 e^x \cos(2x) + c_3 e^x \sin(2x)$

$y_P = Ae^{4x} + Bx^2 + Cx$

$y_P' = 4Ae^{4x} + 2Bx + C$

$y_P'' = 16Ae^{4x} + 2B$

$y_P''' = 64Ae^{4x}$

$64Ae^{4x} - 32Ae^{4x} - 4B + 20Ae^{4x} + 10Bx + 5C = e^{4x} + 2x - 7$

$52A = 1 \qquad 10B = 2 \qquad -4B + 5C = -7$

$A = \frac{1}{52} \qquad B = \frac{1}{5} \qquad C = -\frac{31}{25}$

$y = c_1 + c_2 e^x \cos(2x) + c_3 e^x \sin(2x) + \frac{1}{52} e^{4x} + \frac{1}{5}x^2 - \frac{31}{25}x$

$y(0)=0 \quad 0 = c_1 + c_2 + \frac{1}{52} \quad \boxed{c_1 + c_2 = -\frac{1}{52}}$

$y'(0)=0 \quad y' = c_2 e^x \cos 2x - 2c_2 e^x \sin(2x) + c_3 e^x \sin(2x) + 2c_3 e^x \cos(2x) + \frac{1}{13} e^{4x} + \frac{2}{5}x - \frac{31}{25}$

$0 = c_2 + 2c_3 + \frac{1}{13} - \frac{31}{25} \quad \boxed{c_2 + 2c_3 = \frac{378}{325}}$

$y''(0)=0 \quad y'' = -2c_2 e^x \sin(2x) + c_2 e^x \cos(2x) - 4c_2 e^x \cos(2x) - 2c_2 e^x \sin(2x) + c_3 e^x \sin(2x) + 2c_3 e^x \cos(2x) + 2c_3 e^x \cos(2x) - 4c_3 e^x \sin(2x) + \frac{4}{13} e^{4x} + \frac{2}{5}$

$0 = c_2 - 4c_2 + 2c_3 + 2c_3 + \frac{4}{13} + \frac{2}{5} \quad \boxed{-3c_2 + 4c_3 = \frac{-46}{65}}$

$c_1 = \frac{-313}{500} = -.6260 \qquad c_2 = \frac{986}{1625} \approx .607 \qquad c_3 = \frac{452}{1625} \approx .278$

$$y = \frac{-313}{500} + \frac{986}{1625} e^x \cos(2x) + \frac{452}{1625} e^x \sin(2x) + \frac{1}{52} e^{4x} + \frac{1}{5}x^2 - \frac{31}{25}x$$

2A. $k=3$ $\quad m=\frac{64}{32}=2$ $\qquad 2x''+4x'+3x=0$

$x(0)=\frac{1}{3}$ ft. $\qquad 2\lambda^2+4\lambda+3=0$

$x'(0)=-2$ ft/sec $\qquad \lambda=-1\pm\frac{1}{\sqrt{2}}i$

$$y=c_1e^{-t}\cos\left(\tfrac{1}{\sqrt{2}}t\right)+c_2e^{-t}\sin\left(\tfrac{1}{\sqrt{2}}t\right)$$

$$y'=-\tfrac{1}{\sqrt{2}}c_1e^{-t}\sin\left(\tfrac{1}{\sqrt{2}}t\right)-c_1e^{-t}\cos\left(\tfrac{1}{\sqrt{2}}t\right)$$
$$-c_2e^{-t}\sin\left(\tfrac{1}{\sqrt{2}}t\right)+\tfrac{1}{\sqrt{2}}c_2e^{-t}\cos\left(\tfrac{1}{\sqrt{2}}t\right)$$

$\underline{\frac{1}{3}=c_1}$

$-2=-c_1+\frac{1}{\sqrt{2}}c_2 \Rightarrow \underline{c_2=\frac{-5\sqrt{2}}{3}}$

$-2=-\frac{1}{3}+\frac{1}{\sqrt{2}}c_2$

(6) $\underline{y=\frac{1}{3}e^{-t}\cos\left(\tfrac{1}{\sqrt{2}}t\right)-\frac{5\sqrt{2}}{3}e^{-t}\sin\left(\tfrac{1}{\sqrt{2}}t\right)}$

(2) B. <u>infinite</u>

3. $A\lambda''+2\lambda+9=0$

$4-4A(9)=0$

$-36A=-4$

(2) B. $\underline{A=\frac{1}{9}}$ <u>critically damped</u>

(2) A. $4-36A>0$

$-36A>-4$

$\underline{A<\frac{1}{9}}$

<u>overdamped</u>

4. $x(t)=\sin(4t)-\sqrt{3}\cos(4t)$

(4) A. $A=\sqrt{1+(\sqrt{3})^2}=2$ $\qquad \tan\phi=-\frac{\sqrt{3}}{1}$ $\qquad \left.\begin{matrix}\sin\phi=-\sqrt{3}\\ \cos\phi=1\end{matrix}\right\}$ Q4

$\phi=-\pi/3$

$\underline{x(t)=2\sin(4t-\pi/3)}$

(2) B. $P=\frac{2\pi}{4}=\underline{\frac{\pi}{2}}$

(2) C. $\underline{A=2}$

(2) D. $t=\frac{(3k-2)\pi}{12}$ $\qquad \underline{t=\frac{\pi}{12}}$

5. (6) $0.3\frac{d^2g}{dt^2} + 30\frac{dg}{dt} = 15$

$\frac{d^2g}{dt^2} + 100\frac{dg}{dt} = 50$

$\lambda^2 + 100\lambda = 0$

$\lambda(\lambda+100) = 0$

$\lambda = 0 \quad \lambda = -100$

$g_H = c_1 + c_2 e^{-100t}$

$g_P = Ax$

$g_P' = A \qquad 30A = 15$

$g'' = 0 \qquad A = \frac{1}{2}$

$g(t) = c_1 + c_2 e^{-100t} + \frac{1}{2}t$

$i(t) = -100c_2 e^{-100t} + \frac{1}{2}$

$i(0) = 0 \qquad -100c^2 + \frac{1}{2} = 0$

$c^2 = \frac{1}{200}$

(6) A. $i(t) = -\frac{1}{2}e^{-100t} + \frac{1}{2}$

(2) B. $\lim_{t\to\infty} i(t) = \frac{1}{2}$

6. (7) $2y'' + 8y' + 8y = e^{-2t}\ln(t)$

$y'' + 4y' + 4y = \frac{1}{2}e^{-2t}\ln(t)$

$y(1) = 2$

$y'(1) = -4$

$\lambda^2 + 4\lambda + 4 = 0$

$(\lambda+2)^2 = 0$

$\lambda = -2$

$y_H = c_1 e^{-2t} + c_2 t e^{-2t}$

$$W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t}$$

$u_1' = -\frac{1}{2}t\ln(t)$

$u_1 = \frac{1}{8}t^2 - \frac{1}{4}t^2\ln(t)$

$$W_1 = \begin{vmatrix} 0 & te^{-2t} \\ \frac{1}{2}e^{-2t}\ln t & e^{-2t} - 2te^{-2t} \end{vmatrix} = -\frac{1}{2}te^{-4t}\ln t$$

$u_2' = \frac{1}{2}\ln(t)$

$u_2 = -\frac{1}{2}t + \frac{1}{2}t\ln(t)$

$$W_2 = \begin{vmatrix} e^{-2t} & 0 \\ -2e^{-2t} & \frac{1}{2}e^{-2t}\ln t \end{vmatrix} = \frac{1}{2}e^{-4t}\ln t$$

$y = c_1 e^{-2t} + c_2 te^{-2t} + \left(\frac{1}{8}t^2 - \frac{1}{4}t^2\ln t\right)e^{-2t} + \left(-\frac{1}{2}t + \frac{1}{2}t\ln(t)\right)te^{-2t}$

6. (cont)

$y(1)=2$

$$2 = c_1 e^{-2} + c_2 e^{-2} + \frac{1}{8} e^{-2} - \frac{1}{2} e^{-2}$$

$$2e^2 + \frac{3}{8} = c_1 + c_2$$

$y'(1)=4$

$$y' = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t} - 2\left(\frac{1}{8}t^2 - \frac{1}{4}t^2 \ln t\right)e^{-2t}$$
$$+ e^{-2t}\left(\cancel{\frac{1}{4}t} - \frac{1}{2}t\ln t - \cancel{\frac{1}{4}t}\right) + \left[-\frac{1}{2}t + \frac{1}{2}t\ln(t)\right]\left(-2te^{-2t} + e^{-2t}\right)$$
$$+ te^{-2t}\left(\cancel{-\frac{1}{2}} + \cancel{\frac{1}{2}} + \frac{1}{2}\ln t\right)$$

$$-4 = -2c_1 e^{-2} + c_2 e^{-2} - 2c_2 e^{-2} - \frac{1}{4}e^{-2} + e^{-2} - \frac{1}{2}e^{-2}$$

$$-4 = -2c_1 e^{-2} - c_2 e^{-2} + \frac{1}{4}e^{-2}$$

$$-4 - \frac{1}{4}e^{-2} = -2c_1 e^{-2} - c_2 e^{-2}$$

$$-4e^2 - \frac{1}{4} = -2c_1 - c_2 \qquad c_2 = 2e^2 + \frac{3}{8} - \left(2e^2 - \frac{1}{8}\right) = \frac{1}{2}$$

$$4e^2 + \frac{1}{4} = 2c_1 + c_2$$

$$2e^2 - \frac{1}{8} = c_1$$

$$y = \left(2e^2 - \frac{1}{8}\right)e^{-2t} + \frac{1}{2}te^{-2t} + t^2 e^{-2t}\left(-\frac{3}{8} + \frac{1}{4}\ln t\right) \quad \text{or}$$

$$y = \left(2e^2 - \frac{1}{8}\right)e^{-2t} + \frac{1}{2}te^{-2t} + \left(\frac{1}{8}t^2 - \frac{1}{4}t^2\ln t\right)e^{-2t} + \left(-\frac{1}{2}t + \frac{1}{2}t\ln t\right)te^{-2t}$$

7. (6)

$$\frac{1}{4}g'' + 2g' + 13g = 40\cos(2t) \qquad i(0)=0$$
$$g(0) = 3$$

$$g'' + 8g' + 52g = 160\cos(2t)$$

$$\lambda^2 + 8\lambda + 52 = 0$$
$$\lambda = -4 \pm 6i$$
$$g_H = e^{-4t}\left(c_1\cos(6t) + c_2\sin(6t)\right)$$
$$g_P = A\cos(2t) + B\sin(2t)$$
$$g_P' = -2A\sin(2t) + 2B\cos(2t)$$
$$g_P'' = -4A\cos(2t) - 4B\sin(2t)$$
$$-4A\cos(2t) - 4B\sin(2t) - 16A\sin(2t) + 16B\cos(2t) + 52A\cos(2t) + 52B\sin(2t) = 160\cos(2t)$$

$$48A + 16B = 160$$
$$(-16A + 48B = 0)3$$
$$-48A + 144B = 0$$
$$160B = 160$$
$$B = 1$$
$$A = 3$$

7. (con't)

$$q_p = 3\cos(2t) + \sin(2t)$$

$$q = e^{-4t}\left(c_1\cos(6t) + c_2\sin(6t)\right) + 3\cos(2t) + \sin(2t)$$

$$q(0)=3: \quad 3 = c_1 + 3 \Rightarrow c_1 = 0$$

$$i = -4e^{-4t}\left(c_1\cos(6t) + c_2\sin(6t)\right) + e^{-4t}\left(-6c_1\sin(6t) + 6c_2\cos(6t)\right)$$
$$-6\sin(2t) + 2\cos(2t)$$

$$i(0)=0 \qquad 0 = \cancel{-4c_1} + 6c_2 + 2$$
$$c_2 = \frac{-1}{3}$$

$$q = \frac{-1}{3}e^{-4t}\sin(6t) + 3\cos(2t) + \sin(2t)$$

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